Topic: Graphing combinations of functions

Question: At what angles is the function undefined?

$$\frac{\sin\left(\frac{\theta}{3}+\pi\right)}{\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)+1}$$ in the interval $(-3\pi, 3\pi]$

Answer choices:

A $\theta = \pi/4$ and $\theta = 5\pi/4$

B $\theta = -3\pi/2$ and $\theta = 5\pi/2$

C $\theta = -5\pi/4$ and $\theta = 3\pi/4$

D $\theta = -3\pi/2$, $\theta = \pi/2$, and $\theta = 5\pi/2$

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Solution: B

The functions in the numerator and the denominator,

$$\sin\left(\frac{\theta}{3}+\pi\right)$$

and

$$\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)+1$$

respectively, are defined at all angles θ. Thus the only angles θ at which the given function is undefined are those at which the function in the denominator is equal to 0.

Note that

$$\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)+1=0 \iff \cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)=-1$$

Recall that for any angle α, $\cos\alpha=-1$ if and only if there is some odd integer n such that $\alpha=n\pi$. An integer n is odd if and only if there is some integer m such that $n=2m+1$. Therefore, we'll set

$$\frac{\theta}{2}-\frac{\pi}{4}$$

to $(2m+1)\pi$ and solve it for integers m:

$$\frac{\theta}{2}-\frac{\pi}{4}=(2m+1)\pi$$

The solution consists of using just algebra:

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$$\frac{2(\theta)-1(\pi)}{4}=(2m+1)\pi$$

$$\frac{2\theta-\pi}{4}=(2m+1)\pi$$

$$2\theta-\pi=4(2m+1)\pi$$

$$2\theta=4(2m+1)\pi+\pi$$

$$2\theta=(8m+4)\pi+\pi$$

$$2\theta=(8m+5)\pi$$

$$\theta=\frac{(8m+5)\pi}{2}$$

We need to determine the angles θ in the interval $(-3\pi,3\pi]$ at which $\theta=(8m+5)\pi/2$ for some integer m, so we need to determine the integers m such that

$$-3\pi<\frac{(8m+5)\pi}{2}\leq 3\pi$$

To do this, we'll use only algebra:

$$-3\pi\left(\frac{2}{\pi}\right)<\left[\frac{(8m+5)\pi}{2}\right]\left(\frac{2}{\pi}\right)\leq 3\pi\left(\frac{2}{\pi}\right)$$

$$-6<8m+5\leq 6$$

$$-11<8m\leq 1$$

$$-\frac{11}{8}<m\leq\frac{1}{8}$$

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The only integers m in the interval

$$\left(-\frac{11}{8}, \frac{1}{8}\right]$$

are $m = -1$ and $m = 0$.

To find the corresponding angles θ, we substitute -1 and 0 for m in the equation $\theta = (8m + 5)\pi/2$:

$$m = -1 \qquad \theta = [8(-1) + 5]\left(\frac{\pi}{2}\right) = -3\left(\frac{\pi}{2}\right) = -\frac{3\pi}{2}$$

$$m = 0 \qquad \theta = [8(0) + 5]\left(\frac{\pi}{2}\right) = 5\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$$

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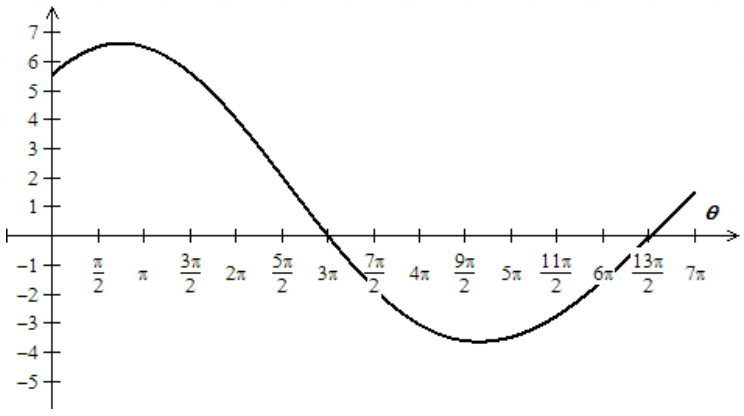
Topic: Graphing combinations of functions

Question: Which is the graph of the function?

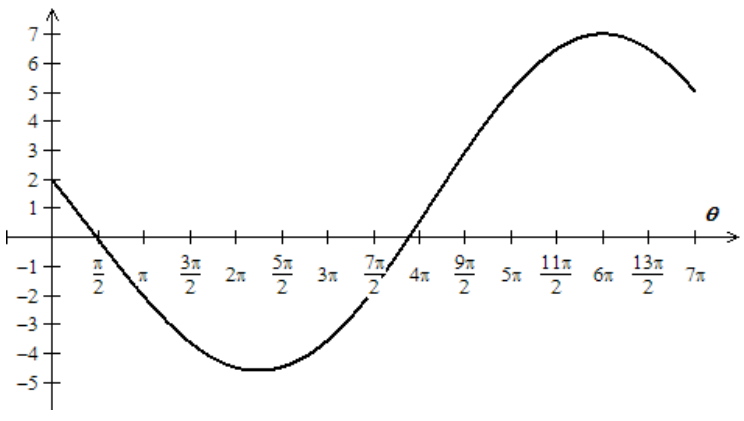
$$5\sin\left(\frac{\theta}{4}+\pi\right)-\left[\cos\left(\frac{\theta}{3}+\pi\right)-1\right] \text{ on the interval } [0,7\pi)$$

Answer choices:

A



B



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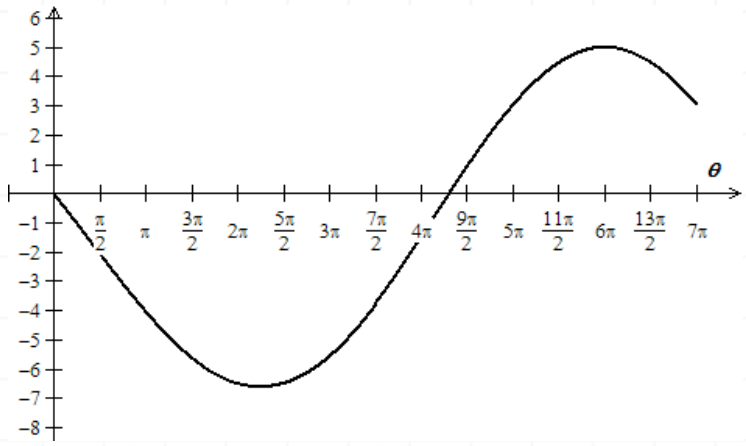
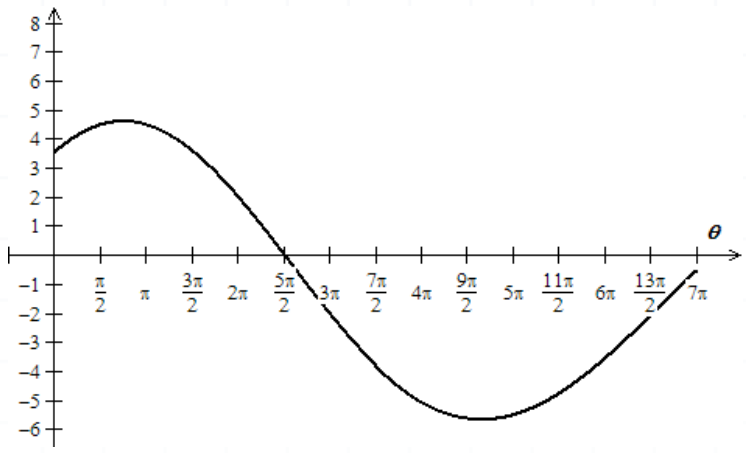
C



D



Solution: B

Notice that at $\theta = 0$ the values of the dependent variable in the four curves are all different, so let's compute the value of the function

$$5\sin\left(\frac{\theta}{4}+\pi\right)-\left[\cos\left(\frac{\theta}{3}+\pi\right)\ -1\right]$$

at $\theta = 0$.

Well, at $\theta = 0$,

$$5\sin\left(\frac{\theta}{4}+\pi\right)=5\sin\left(\frac{0}{4}+\pi\right)=5\sin\pi=0$$

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and

$$\cos\left(\frac{\theta}{3}+\pi\right)-1=\cos\left(\frac{0}{3}+\pi\right)-1=\cos\pi-1=-1-1=-2$$

Therefore, at $\theta=0$,

$$5\sin\left(\frac{\theta}{4}+\pi\right)-\left[\cos\left(\frac{\theta}{3}+\pi\right)-1\right]=0-(-2)=2$$

This verifies that the curve shown in answer choice B is correct, because it's the only curve in which the value of the dependent variable at $\theta=0$ is equal to 2.

Let's graph the sine function and the cosine function whose difference is the function given in answer choice B, and their difference, so that we can see the genesis of the graph of the difference.

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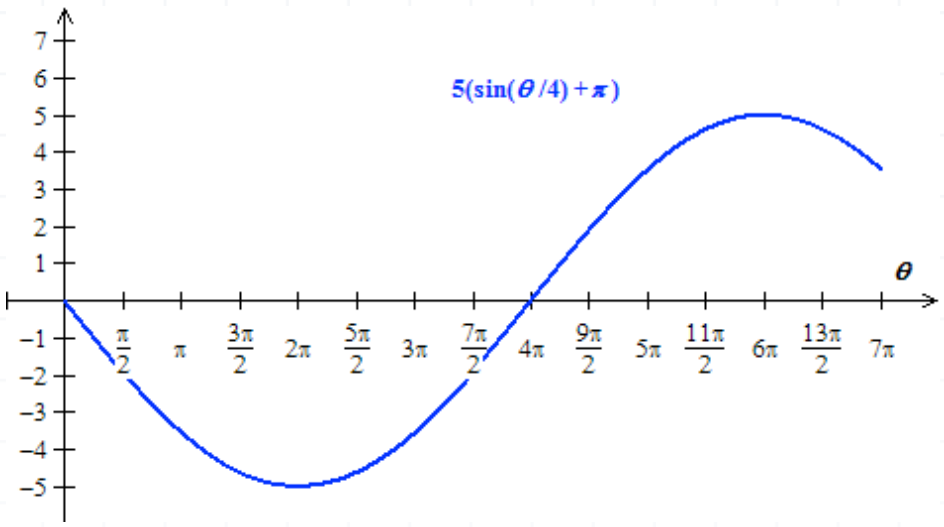

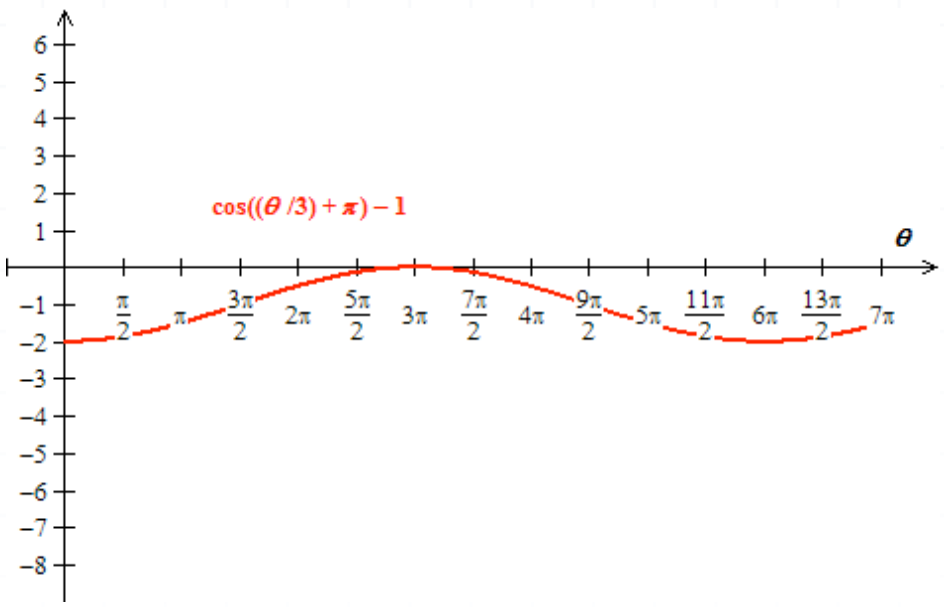

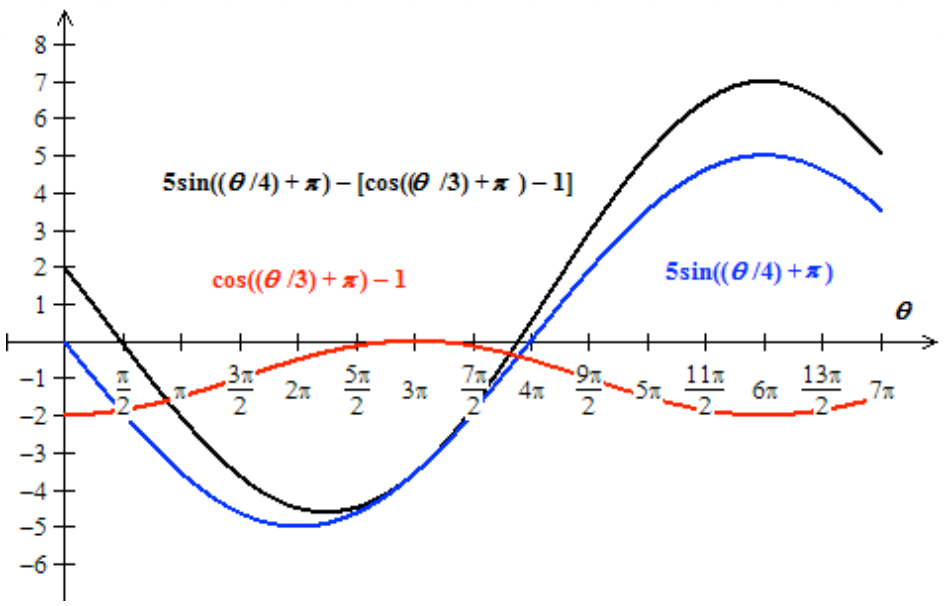

In graphing the difference, we need to pay close attention to signs. Notice that

$$-\left[\cos\left(\frac{\theta}{3}+\pi\right)-1\right]=-\cos\left(\frac{\theta}{3}+\pi\right)-(-1)=-\cos\left(\frac{\theta}{3}+\pi\right)+1$$

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If we were to mistakenly graph

$$-\left[\cos\left(\frac{\theta}{3}+\pi\right)\right]-1$$

instead of

$$-\cos\left(\frac{\theta}{3}+\pi\right)+1$$

we would get the graph that's shifted vertically by -2 units (i.e., shifted 2 units downwards) with respect to the correct graph, thereby leading us to conclude that answer choice C is correct.

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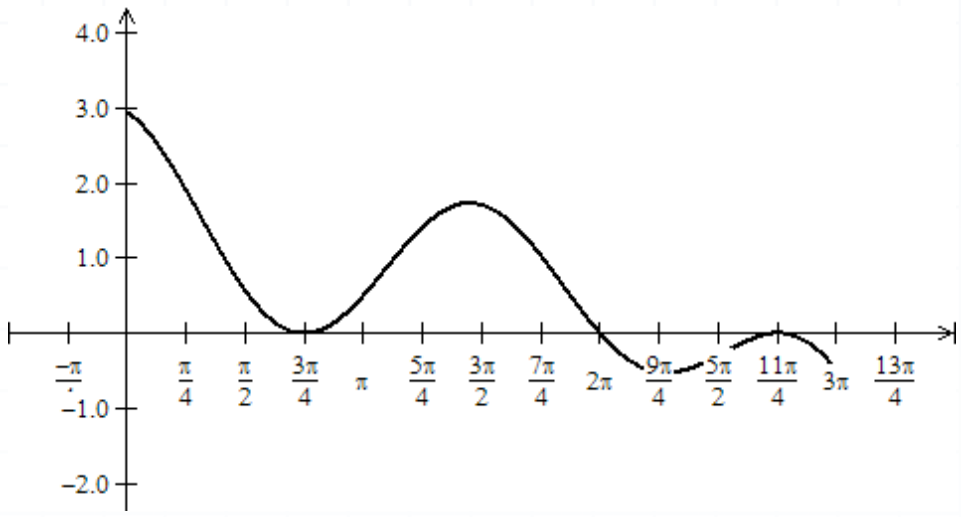
Topic: Graphing combinations of functions

Question: The following curve is the graph, over the interval $[0,3\pi)$, of a function which is the product of a sine function and a cosine function. Which product does the graph represent.



Answer choices:

A $\left(3\cos\left(\frac{2\theta}{3}-\frac{\pi}{4}\right)\right)\left(\sin\left(\theta-\frac{\pi}{3}\right)+1\right)$

B $\left(\sin\left(\frac{2\theta}{3}+\frac{\pi}{6}\right)\right)\left(4\cos\left(\theta-\frac{\pi}{2}\right)-1\right)$

C $\left(2\cos\left(\frac{\theta}{3}-\frac{\pi}{6}\right)\right)\left(\sin\left(\theta+\frac{3\pi}{4}\right)+1\right)$

D $\left(\sin\left(\theta+\frac{2\pi}{3}\right)\right)\left(2\cos\left(3\theta+\frac{\pi}{4}\right)-1\right)$

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Solution: C

Let's consider the angles θ at which the function that corresponds to the given graph is equal to 0. Inspection of the graph tells us that in the interval $[0,3\pi)$, those angles are $3\pi/4$, 2π, and $11\pi/4$.

Now we'll show that the function given in answer choice C,

$$\left(2\cos\left(\frac{\theta}{3}-\frac{\pi}{6}\right)\right)\left(\sin\left(\theta+\frac{3\pi}{4}\right)+1\right)$$

is indeed equal to 0 at all three of the angles $3\pi/4$, 2π, and $11\pi/4$.

First, we'll show that at $\theta = 3\pi/4$, the second function in the product of functions given in answer choice C,

$$\sin\left(\theta+\frac{3\pi}{4}\right)+1$$

is equal to 0:

$$\theta=\frac{3\pi}{4} \quad \rightarrow \quad \theta+\frac{3\pi}{4}=\frac{3\pi}{4}+\frac{3\pi}{4}=\frac{3\pi}{2}$$

$$\rightarrow \quad \sin\left(\theta+\frac{3\pi}{4}\right)+1=\sin\left(\frac{3\pi}{2}\right)+1=-1+1=0$$

Next, we'll show that at $\theta = 2\pi$, the first function in the product of functions given in answer choice C,

$$2\cos\left(\frac{\theta}{3}-\frac{\pi}{6}\right)$$

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is equal to 0:

$$\theta = 2\pi \quad \rightarrow \quad \frac{\theta}{3} - \frac{\pi}{6} = \frac{2\pi}{3} - \frac{\pi}{6} = \frac{2(2\pi) - 1(\pi)}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$$

$$\rightarrow \quad 2\cos\left(\frac{\theta}{3} - \frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{2}\right) = 2(0) = 0$$

Finally, we'll show that at $\theta = 11\pi/4$, the second function in the product of functions given in answer choice C,

$$\sin\left(\theta + \frac{3\pi}{4}\right) + 1$$

is equal to 0:

$$\theta = \frac{11\pi}{4} \rightarrow \quad \theta + \frac{3\pi}{4} = \frac{11\pi}{4} + \frac{3\pi}{4} = \frac{14\pi}{4} = \frac{8\pi}{4} + \frac{6\pi}{4} = 2\pi + \frac{3\pi}{2}$$

$$\rightarrow \quad \sin\left(\theta + \frac{3\pi}{4}\right) + 1 = \sin\left(2\pi + \frac{3\pi}{2}\right) + 1 = \sin\left(\frac{3\pi}{2}\right) + 1 = -1 + 1 = 0$$

Now we'll test the function given in each of the other three answer choices, to see if they're equal to 0 at $\theta = 3\pi/4$. In each case, we'll show that neither of the two functions in the given product is equal to 0 at $\theta = 3\pi/4$.

Setting θ to $3\pi/4$ in the function given in answer choice A,

$$\left(3\cos\left(\frac{2\theta}{3} - \frac{\pi}{4}\right)\right)\left(\sin\left(\theta - \frac{\pi}{3}\right) + 1\right)$$

we have the following:

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$$\theta = \frac{3\pi}{4} \quad \rightarrow \quad \frac{2\theta}{3} - \frac{\pi}{4} = \frac{2\left(\frac{3\pi}{4}\right)}{3} - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{2(\pi) - 1(\pi)}{4} = \frac{\pi}{4}$$

$$\rightarrow \quad 3\cos\left(\frac{2\theta}{3} - \frac{\pi}{4}\right) = 3\cos\left(\frac{\pi}{4}\right) = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2} \neq 0$$

$$\theta = \frac{3\pi}{4} \quad \rightarrow \quad \theta - \frac{\pi}{3} = \frac{3\pi}{4} - \frac{\pi}{3} = \frac{3(3\pi) - 4(\pi)}{12} = \frac{5\pi}{12}$$

$$\rightarrow \quad \sin\left(\theta - \frac{\pi}{3}\right) = \sin\left(\frac{5\pi}{12}\right) \neq -1$$

$$\rightarrow \quad \sin\left(\theta - \frac{\pi}{3}\right) + 1 \neq 0$$

Since neither of the functions of the product given in answer choice A is equal to 0 at $\theta = 3\pi/4$, we can eliminate answer choice A.

Setting θ to $3\pi/4$ in the function given in answer choice B,

$$\left(\sin\left(\frac{2\theta}{3} + \frac{\pi}{6}\right)\right)\left(4\cos\left(\theta - \frac{\pi}{2}\right) - 1\right)$$

we have the following:

$$\theta = \frac{3\pi}{4} \quad \rightarrow \quad \frac{2\theta}{3} + \frac{\pi}{6} = \frac{2\left(\frac{3\pi}{4}\right)}{3} + \frac{\pi}{6} = \frac{\pi}{2} + \frac{\pi}{6} = \frac{3(\pi) + 1(\pi)}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$\rightarrow \quad \sin\left(\frac{2\theta}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{3}\right) \neq 0$$

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$$\theta = \frac{3\pi}{4} \quad \rightarrow \quad \theta - \frac{\pi}{2} = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{1(3\pi) - 2(\pi)}{4} = \frac{\pi}{4}$$

$$\rightarrow \quad 4\cos\left(\theta - \frac{\pi}{2}\right) = 4\cos\left(\frac{\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2} \neq 1$$

$$\rightarrow \quad 4\cos\left(\theta - \frac{\pi}{2}\right) - 1 \neq 0$$

Neither of the functions of the product given in answer choice B is equal to 0 at $\theta = 3\pi/4$, so we can eliminate answer choice B.

Setting θ to $3\pi/4$ in the function given in answer choice D,

$$\left(\sin\left(\theta + \frac{2\pi}{3}\right)\right)\left(2\cos\left(3\theta + \frac{\pi}{4}\right) - 1\right)$$

we have the following:

$$\theta = \frac{3\pi}{4} \quad \rightarrow \quad \theta + \frac{2\pi}{3} = \frac{3\pi}{4} + \frac{2\pi}{3} = \frac{3(3\pi) + 4(2\pi)}{12} = \frac{17\pi}{12}$$

$$\rightarrow \quad \sin\left(\theta + \frac{2\pi}{3}\right) = \sin\left(\frac{17\pi}{12}\right) \neq 0$$

$$\theta = \frac{3\pi}{4} \quad \rightarrow \quad 3\theta + \frac{\pi}{4} = 3\left(\frac{3\pi}{4}\right) + \frac{\pi}{4} = \frac{9\pi}{4} + \frac{\pi}{4} = \frac{10\pi}{4} = \frac{8\pi}{4} + \frac{2\pi}{4} = 2\pi + \frac{\pi}{4}$$

$$\rightarrow \quad 2\cos\left(3\theta + \frac{\pi}{4}\right) = 2\cos\left(2\pi + \frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \neq 1$$

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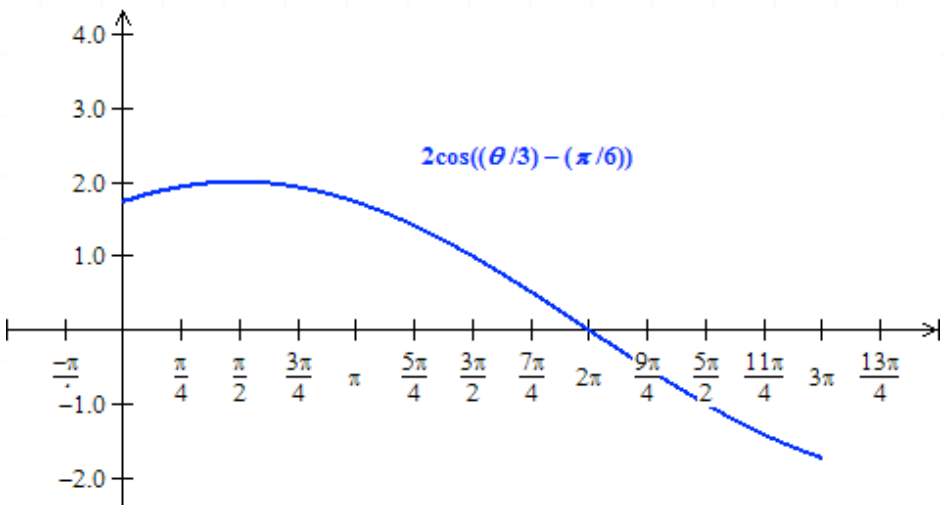
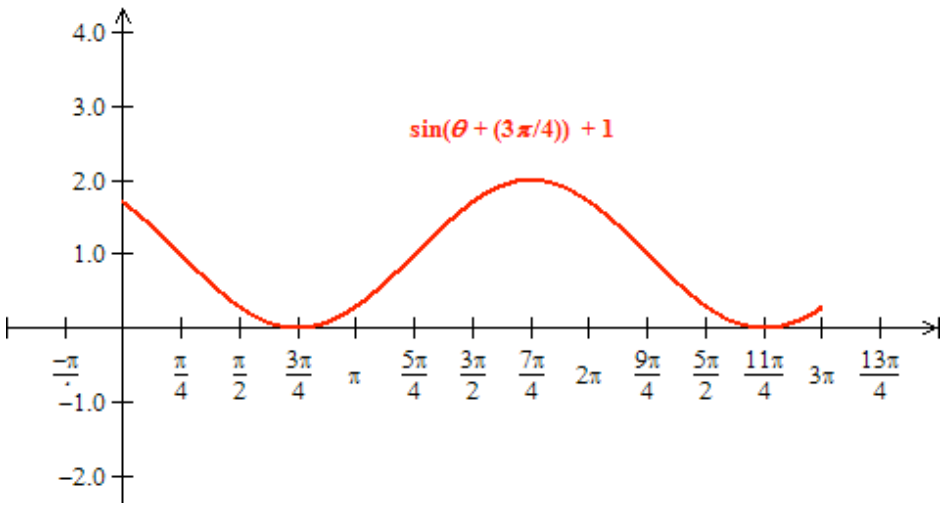
$$\rightarrow \quad 2\cos\left(3\theta + \frac{\pi}{4}\right) - 1 \neq 0$$

Neither of the functions of the product given in answer choice D is equal to 0 at $\theta = 3\pi/4$, so we can eliminate answer choice D.

Let's graph the cosine function and the sine function whose product is the function given in answer choice C, and their product, so that we can see the genesis of the graph of their product.





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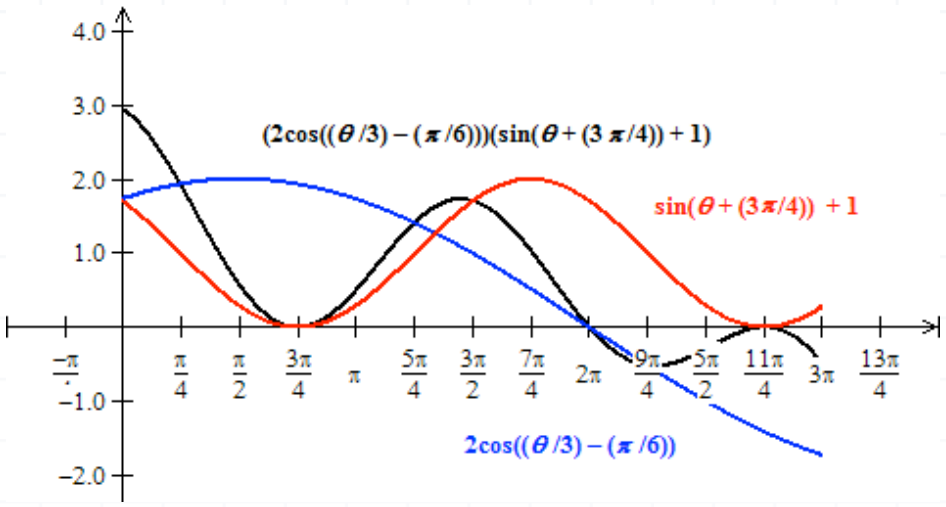

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